Guiding text is text that guides the reader by
describing the structure of the logic or
signaling the structure of the text.

The Existence of $\sqrt{2}$

The importance of the Supremum Property lies in the fact that it guarantees the existence of real numbers under certain hypotheses. We shall make use of it in this way many times. At the moment, we shall illustrate this use by proving the existence of a positive real number x such that $x^2 = 2$; that is, the positive square root of 2. It was shown earlier (see Theorem 2.1.4) that such an x cannot be a rational number; thus, we will be deriving the existence of at least one irrational number.

2.4.7 Theorem *There exists a positive real number x such that $x^2 = 2$.*

Proof. Let $S := \{s \in \mathbb{R} : 0 \le s, s^2 < 2\}$. Since $1 \in S$, the set is not empty. Also, S is bounded above by 2, because if $t > 2$, then $t^2 > 4$ so that $t \notin S$. Therefore the Supremum Property implies that the set S has a supremum in $\mathbb{R}$, and we let $x := \sup S$. Note that $x > 1$.

start list → We will prove that $x^2 = 2$ by ruling out the other two possibilities: $x^2 < 2$ and $x^2 > 2$.

start list item → First assume that $x^2 < 2$. We will show that this assumption contradicts the fact that $x = \sup S$ by finding an $n \in \mathbb{N}$ such that $x + 1/n \in S$, thus implying that x is not an upper bound for S. To see how to choose n, note that $1/n^2 \le 1/n$ so that

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \le x^2 + \frac{1}{n}(2x + 1).$$

Hence if we can choose n so that

$$\frac{1}{n}(2x + 1) < 2 - x^2,$$

then we get $(x + 1/n)^2 < x^2 + (2 - x^2) = 2$. By assumption we have $2 - x^2 > 0$, so that $(2 - x^2)/(2x + 1) > 0$. Hence the Archimedean Property (Corollary 2.4.5) can be used to obtain $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{2 - x^2}{2x + 1}.$$

These steps can be reversed to show that for this choice of n we have $x + 1/n \in S$, which contradicts the fact that x is an upper bound of S. Therefore we cannot have $x^2 < 2$. ← end list item

start list item → Now assume that $x^2 > 2$. We will show that it is then possible to find $m \in \mathbb{N}$ such that $x - 1/m$ is also an upper bound of S, contradicting the fact that $x = \sup S$. To do this, note that

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}.$$

Hence if we can choose m so that

$$\frac{2x}{m} < x^2 - 2,$$

end list item → Therefore we cannot have $x^2 > 2$.

end list → Since the possibilities $x^2 < 2$ and $x^2 > 2$ have been excluded, we must have $x^2 = 2$.

Q.E.D.

then $(x - 1/m)^2 > x^2 - (x^2 - 2) = 2$. Now by assumption we have $x^2 - 2 > 0$, so that $(x^2 - 2)/2x > 0$. Hence, by the Archimedean Property, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \frac{x^2 - 2}{2x}.$$

These steps can be reversed to show that for this choice of m we have $(x - 1/m)^2 > 2$. Now if $s \in S$, then $s^2 < 2 < (x - 1/m)^2$, whence it follows from 2.1.13(a) that $s < x - 1/m$. This implies that $x - 1/m$ is an upper bound for S, which contradicts the fact that $x = \sup S$. Therefore we cannot have $x^2 > 2$.

Since the possibilities $x^2 < 2$ and $x^2 > 2$ have been excluded, we must have $x^2 = 2$.

Q.E.D.

By slightly modifying the preceding argument, the reader can show that if $a > 0$, then there is a unique $b > 0$ such that $b^2 = a$. We call b the **positive square root** of a and denote it by $b = \sqrt{a}$ or $b = a^{1/2}$. A slightly more complicated argument involving the binomial theorem can be formulated to establish the existence of a unique **positive *n*th root** of a, denoted by $\sqrt[n]{a}$ or $a^{1/n}$, for each $n \in \mathbb{N}$.

Remark If in the proof of Theorem 2.4.7 we replace the set S by the set of rational numbers $T := \{r \in \mathbb{Q} : 0 \le r, r^2 < 2\}$, the argument then gives the conclusion that $y := \sup T$ satisfies $y^2 = 2$. Since we have seen in Theorem 2.1.4 that y cannot be a rational number, it follows that the set T that consists of rational numbers does not have a supremum belonging to the set $\mathbb{Q}$. Thus the ordered field $\mathbb{Q}$ of rational numbers does *not* possess the Completeness Property.

Text is from pages 41-42 of Robert G Bartle and Donald R Sherbert's Introduction to Real Analysis, 3rd Edition.